Expansion of real numbers by algebraic numbers

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In this talk we discuss fractional parts of powers of algebraic numbers. Koksma proved that when we take any real $\alpha > 1$ then the sequence $(\xi\alpha^n)_{n=0}^{\infty}$ is uniformly distributed modulo 1 for almost all ξ. Similarly if ξ is a non-zero real number, the sequence $(\xi\alpha^n)_{n=0}^{\infty}$ is uniformly distributed modulo 1 for almost all $\alpha > 1$. On the contrary, it is difficult to determine whether the sequence $(\xi\alpha^n)_{n=0}^{\infty}$ is uniformly distributed modulo 1 for fixed ξ and α. Moreover, it isn't generally known that the fractional parts of $(\xi\alpha^n)_{n=0}^{\infty}$ is dense on the torus $\mathbb{R}/\mathbb{Z}$.

When N is a natural number with $N \geq 2$, the fractional parts of ξN^n is determined by the N-ary expansion of ξ. Mahler introduced the "3/2-ary" expansion to estimate the values $\xi(3/2)^n$ modulo 1. He studied the set of Z-numbers by using this numerical system.

We now introduce the "α-ary" expansion of ξ, where $\alpha > 1$ is an algebraic number whose Galois conjugates have absolute values greater than 1. We can express the integer parts and fractional parts of $\xi\alpha^n$ by this system. When α is not a natural number, digits of "α-ary" expansion have some interesting properties. By studying the patterns of digits, we can estimate the fractional parts of $\xi\alpha^n$.